GENERAL MATHEMATICS | UNIT 2

Operations on Functions

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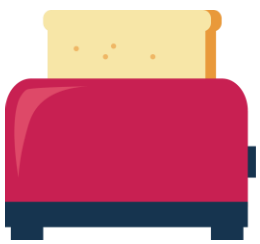
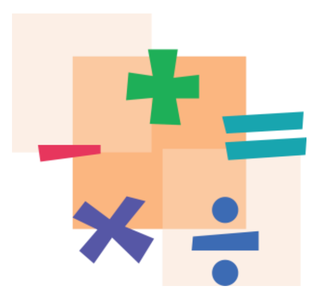
UNIT 2

Operations on Functions

Mathematics is everywhere from computers to aircrafts, from the kitchen to the most sophisticated laboratories. Math is evident in all fields of sciences, and its application to daily life is abundant. Hence, it is imperative to have a good understanding of one of the most important concepts in mathematics, *functions*.



We define a *function* as a relation between given sets of elements. These relationships are evident either numerically, algebraically, or through numerous models of real-life situations. Everyday objects like washing machine, oven toaster, computer, and smartphone in your pockets revolve around the idea of relations and functions. The underlying concepts in these machines may be technological, but ultimately they are mathematical in nature.



In journey through Mathematics, you have encountered linear, quadratic, and even polynomial functions. These functions, like real numbers, can also be added, subtracted, multiplied, or divided with each other.

This unit will help you learn the four different operations on functions, along with a special operation known as *composition*.

Test Your Prerequisite Skills

- Adding, subtracting, multiplying, and dividing polynomials
- Evaluating functions
- Factoring polynomials
- Solving word problems involving functions and polynomials

Before you get started, answer the following items on a separate sheet of paper. This will help you assess your prior knowledge and practice some skills that you will need in studying the lessons in this unit. Show your complete solution.

1. Perform the indicated operations.
   a. $(3 - mx - 3x^2) - (-mx - 2x^2 - 4)$
   b. $(x + y)(x - y)$
   c. $(x + 3)(x - 2)$
   d. $\frac{-156x^2y^4z^3}{26x^2y^2z^2}$
   e. $(7x + 12 + x^2) \div (x + 3)$

2. Factor the following expressions completely.
   a. $36x^2y^2 - 30xy^3$
   b. $s^2 - 81$
   c. $t^3 - 27$
   d. $64 + 34p + p^2$
   e. $x^2 + 5x + 6$

3. Evaluate the following functions based on the given value of x.
   a. $f(x) = x^2 - 4x + 5;\ x = -1$
   b. $f(x) = x^3 - x^2 - 5x + 12;\ x = 3$
   c. $g(x) = x^4 - 4x^2 - x + 5;\ x = -2$
   d. $h(x) = 4(-x^4 + x - 3x^2);\ x = 2$
   e. $h(x) = \frac{x^2+3x-2}{x+5};\ \ x = 5$

4. A farmer is given a rectangular piece of land that measures $(x+3)$ by $(2x^2+1)$ meters. What is the current area of the land owned by the farmer? If the dimensions were doubled, what would be the new area?

Objectives

At the end of this unit, you should be able to

- perform addition, subtraction, multiplication, division, and composition of functions; and
- solve problems involving functions.

Lesson 1: Addition and Subtraction of Functions

Warm Up!

Let's Roll!

Materials Needed: colored papers, pair of dice, pen, *cartolina*, marker

Instructions:

1. This activity can be played by pair or by group.
2. Your teacher will prepare 6 different sets of algebraic expressions to be written on colored papers.
3. Each expression will be assigned a number from one to six.
4. You will be given time to roll the pair of dice 5 times.
5. Record the results of your rolls for each pair and write the corresponding expression.

6. For each pair of expressions, roll a dice again. If the resulting number is odd, add the expressions. However, if the result is even, subtract the expressions.
7. Perform the indicated operation based on your roll and show your solution by writing it on your paper or *cartolina*.
8. Your teacher will call volunteers to explain their work in front of the class.

Learn about It!

Let's say that the expressions you have encountered in the *Warm Up!* activity denote a function. As a recall, we define a **function** as a special type of relation wherein each input only has one output.

Functions can be used to model situations in real life. Since they are algebraic in nature, the basic operations can be applied as what you have seen in *Warm Up!*.

We can denote the first expression from your first roll as $f(x)$ while the second as $g(x)$. When given two functions $f(x)$ and $g(x)$, their sum is denoted by $(f+g)(x)$. This is defined by $(f+g)(x) = f(x) + g(x)$.

By the definition of the sum of two functions, we just add the two given expressions for $f(x)$ and $g(x)$.

Given the functions $f(x) = 5x + 2$ and $g(x) = 8 + 6x - 2x^2$, let us find $(f+g)(x)$.

$(f+g)(x) = f(x) + g(x)$	
$(f+g)(x) = (5x+2) + (8+6x-2x^2)$	Substitute the given functions.
$(f+g)(x) = (2+8) + (5x+6x) - 2x^2$	Group similar terms together.
$(f+g)(x) = 10 + 11x - 2x^2$	Combine similar terms.

Hence, $(f+g)(x) = 10 + 11x - 2x^2$ or $(f+g)(x) = -2x^2 + 11x + 10$.

On the other hand, if we are given two functions $f(x)$ and $g(x)$, then we can find their difference, denoted by $(f-g)(x)$ as $(f-g)(x) = f(x) - g(x)$.

Since a function is being subtracted from another function, every term of the subtrahend is being subtracted from the minuend. Recall that when we subtract algebraic expressions, we copy the minuend, then change the operation from subtraction to addition, and we change the signs of all the terms in the subtrahend.

Given the functions $f(x) = 5x + 2$ and $g(x) = 8 + 6x - 2x^2$, let us find $(f-g)(x)$.

$(f-g)(x) = f(x) - g(x)$	
$(f-g)(x) = (5x+2) - (8 + 6x - 2x^2)$	Substitute the given functions.
$(f-g)(x) = (5x+2) + (-8 - 6x + 2x^2)$	Change the operation from subtraction to addition. Change the signs of all the terms in the subtrahend.
$(f-g)(x) = (2-8) + (5x - 6x) + 2x^2$	Group similar terms together.
$(f-g)(x) = -6 - x + 2x^2$	Combine similar terms.

Hence, $(f-g)(x) = -6 - x + 2x^2$ or $(f-g)(x) = 2x^2 - x - 6$.

Let us study some other examples below to better understand addition and subtraction of functions.

Let's Practice!

Example 1: Given the functions $g(x) = 8 + 6x - 2x^2$ and $h(x) = 4(-x^4 + x - 3x^2)$, find $(g+h)(x)$.

Solution: We find $(g+h)(x)$ by adding the two given expressions for $g(x)$ and $h(x)$.

$(g+h)(x) = g(x) + h(x)$	
$(g+h)(x) = (8 + 6x - 2x^2) + 4(-x^4 + x - 3x^2)$	Substitute the given functions.
$(g+h)(x) = (8 + 6x - 2x^2) + (-4x^4 + 4x - 12x^2)$	Distribute 4 to all of the terms inside the second parentheses.
$(g+h)(x) = (-4x^4) + (-2x^2 - 12x^2) + (4x + 6x) + (8)$	Group similar terms together.
$(g+h)(x) = -4x^4 - 14x^2 + 10x + 8$	Combine similar terms.

Try It Yourself!

Given the functions $g(x) = x^3 - x^4 + 6x - 3$ and $h(x) = 2(-x^4 - 2x + 4x^2)$, find $(g+h)(x)$.

Example 2: What is $(f-g)(4)$ if $f(x) = 5x + 2$ and $g(x) = 15x + 6$?

Solution:

Step 1: Subtract the two given expressions for $f(x)$ and $g(x)$.

$(f-g)(x) = f(x) - g(x)$	
$(f-g)(x) = (5x + 2) - (15x + 6)$	Substitute the given functions.
$(f-g)(x) = (5x + 2) + (-15x - 6)$	Change the operation from subtraction to addition. Change the signs of all the terms in the subtrahend.
$(f-g)(x) = (2 - 6) + (5x - 15x)$	Group similar terms together.
$(f-g)(x) = -4 - 10x$	Combine similar terms.

Step 2: Substitute $x = 4$ to the resulting function.

$$(f - g)(4) = -4 - 10(4)$$
$$(f - g)(4) = -48$$

Try It Yourself!

What is $(f - g)(3)$ if $f(x) = 5x^2 - 3$ and $g(x) = x^2 - 15x + 6$

Example 3: Find $(f + g)(x)$, given $f(x) = \frac{x+7}{x+1}$ and $g(x) = \frac{x-2}{x+3}$.

Solution:

Step 1: Let us first look for the LCD of the two functions. The LCD is $(x + 1)(x + 3)$ or $x^2 + 4x + 3$.

Step 2: To find $(f + g)(x)$, we solve for $(f + g)(x) = f(x) + g(x)$.

$$(f + g)(x) = f(x) + g(x)$$

$$(f + g)(x) = \frac{x+7}{x+1} + \frac{x-2}{x+3}$$ Substitute the given functions.

$$(f + g)(x) = \frac{(x+7)(x+3)+(x-2)(x+1)}{x^2+4x+3}$$ Add the functions using the LCD.

$$(f + g)(x) = \frac{(x^2+10x+21)+(x^2-x-2)}{x^2+4x+3}$$ Simplify the numerator.

$$(f + g)(x) = \frac{(x^2+x^2)+(10x-x)+(21-2)}{x^2+4x+3}$$ Combine similar terms in the numerator.

$$(f + g)(x) = \frac{2x^2+9x+19}{x^2+4x+3}$$

Try It Yourself!

Find $(f - g)(x)$, given $f(x) = \frac{x+7}{x+1}$ and $g(x) = \frac{x-2}{x+3}$.

Real-World Problems

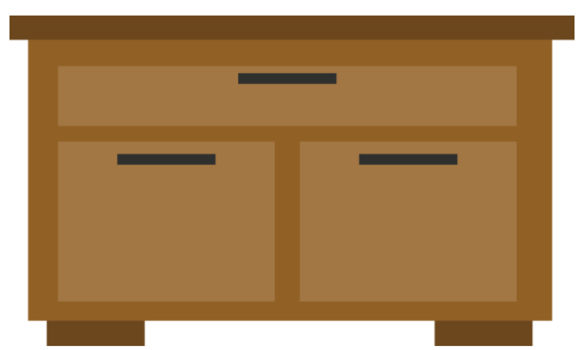
Example 4: The HR Department informed Karl that his starting salary as a car sales agent will be $f(x) = 10x^2 - 5x + 12$ pesos. He will receive $g(x) = 2x^2 + 3x - 5$ pesos as incentive if he is able to meet the company quota for six months. What will be his new salary after six months, assuming that he meets the quota?

Solution: To find his new salary after six months, add his starting salary and incentive.

$(f + g)(x) = f(x) + g(x)$	
$(f + g)(x) = (10x^2 - 5x + 12) + (2x^2 + 3x - 5)$	Substitute the given functions.
$(f + g)(x) = (10x^2 + 2x^2) + (-5x + 3x) + (12 - 5)$	Group similar terms together.
$(f + g)(x) = 12x^2 - 2x + 7$	Combine similar terms.

Hence, Karl's monthly salary will be $(12x^2 - 2x + 7)$ pesos if he will be able to meet the quota after six months.

Example 5: Christian is making a cabinet. He bought a slab of wood that is $f(x) = 9x^2 - 3x + 2$ feet long. What is the length of the wood after $g(x) = 2x^2 + 5x - 4$ feet have been cut off?



Solution: Let $f(x) = 9x^2 - 3x + 2$ be the original length of the wood and $g(x) = 2x^2 + 5x - 4$ be the part that was cut off. To find the length of the remaining wood, subtract the measure of the part of the wood that was cut off from the original length.

$(f - g)(x) = f(x) - g(x)$	
$(f - g)(x) = (9x^2 - 3x + 2) - (2x^2 + 5x - 4)$	Substitute the given functions.
$(f - g)(x) = (9x^2 - 3x + 2) + (-2x^2 - 5x + 4)$	Change the operation from subtraction to addition. Change the signs of all the terms in the subtrahend.
$(f - g)(x) = (9x^2 - 2x^2) + (-3x - 5x) + (2 + 4)$	Group similar terms together.
$(f - g)(x) = 7x^2 - 8x + 6$	Combine similar terms.

Therefore, the length of the remaining wood is $(7x^2 - 8x + 6)$ feet.

Try It Yourself!

Johnny is making a balance sheet for his monthly allowance. He already had two entries on his debit and three entries on his credit. If his debit entries are x^5 and $4x^4$, while the entries for his credit are $x^3 + 3$, $-x^2$, and x^3, express his total allowance $(d - c)(5)$ in terms of his total debit and total credit defined by $d(x)$ and $c(x)$, respectively. (Assume that all values are in pesos.)

Check Your Understanding!

1. Consider the following functions:

   $a(x) = x^2 + 7x - 8$
   $b(x) = \frac{x+7}{x+2}$
   $c(x) = \frac{2x-3}{x+2}$
   $d(x) = x + 3$
   $e(x) = 2x - 7$
   $f(x) = x^2 + 5x + 4$
   $g(x) = \frac{x-2}{x-3}$
   $h(x) = \frac{x-1}{x+1}$
   $i(x) = x^3 - 3x^2 - 2x + 8$

   Find:

   a. $(d + e)(x)$
   b. $(d + f)(-3)$
   c. $(e + a)(x)$
   d. $(f - e)(x)$
   e. $(b + c)(2)$
   f. $(b - c)(-1)$
   g. $(f - d)()$
   h. $(g + h)(x)$
   i. $(i - a)(1)$

2. Consider the following functions:

   $p(x) = x^2 - 2x + 1$
   $q(x) = \frac{2x+1}{x-1}$
   $r(x) = x^3 - x^2 + 3x + 1$
   $s(x) = x^2 + 3x + 5$
   $t(x) = 2x + 1$
   $u(x) = 2x - 1$
   $v(x) = x^3 - 2x - 1$

   a. Express $f_1(x) = x^2 + 2$ as a sum or difference of any two given functions.
   b. Express $f_2(x) = x^3$ as a sum or difference of any two given functions.
   c. Express $f_3(x) = -x^2 + 4x$ as a sum or difference of any two given functions.
   d. Express $f_4(x) = -5x - 4$ as a sum or difference of any two given functions.
   e. Express $f_5(x) = x^3 - x^2 - 5x - 6$ as a sum or difference of any two given functions.

3. Joanna is making a cake. The flour mixture is $f(x) = \frac{x+3}{x-5}$ grams while the other ingredients are $i(x) = x^2 - 25$ grams. What is the total weight $m(x)$ of the mixture if the flour is added to the other ingredients?

Lesson 2: Multiplication of Functions

Warm Up!

Look, Pair, Share!

Materials Needed: colored papers, paper and pen

Instructions:

1. This activity can be played by the whole class.
2. Your teacher has hidden different colored papers under your armchair containing algebraic expressions.
3. When your teacher gives the signal, collect the colored paper assigned to you and look for someone with the same colored paper. Once you are with your partner, multiply the two expressions written on the paper.
4. The first five pairs to give the correct answer win.

Learn about It!

As what we have discussed previously, functions can be added or subtracted from each other. Furthermore, aside from these two operations, they can also be multiplied.

If we denote the expression that you got from the activity in *Warm Up!* with $Y(x)$ and the expression that your classmate got with $C(x)$, then we will have two functions. To find their product, we multiply the two expressions.

When given two functions, say $Y(x)$ and $C(x)$, or $f(x)$ and $g(x)$, their product, denoted by $(f \cdot g)(x)$, is the function defined by $(f \cdot g)(x) = f(x) \cdot g(x)$. By the definition of the product of two functions, we just multiply the two given expressions for $f(x)$ and $g(x)$.

Let us study the example given below.

Given the functions $f(x) = x^2$ and $g(x) = 5x + 2$, find $(f \cdot g)(x)$.

$(f \cdot g)(x) = f(x) \cdot g(x)$	
$(f \cdot g)(x) = (x^2)(5x + 2)$	Substitute the given functions.
$(f \cdot g)(x) = (x^2)(5x) + (x^2)(2)$	Multiply the expressions.
$(f \cdot g)(x) = 5x^3 + 2x^2$	Simplify.

Let us study other examples below to better understand multiplication of functions.

Let's Practice!

Example 1: What is $(f \cdot g)(x)$ if $f(x) = 5x + 2$ and $g(x) = 8 + 6x - 2x^2$?

Solution: Multiply the two given expressions for $f(x)$ and $g(x)$.

$(f \cdot g)(x) = f(x) \cdot g(x)$	
$(f \cdot g)(x) = (5x + 2)(8 + 6x - 2x^2)$	Substitute the given functions.
$(f \cdot g)(x) = (5x)(8) + (5x)(6x) + (5x)(-2x^2)$ $+(2)(8) + (2)(6x) + (2)(-2x^2)$	Multiply the expressions.
$(f \cdot g)(x) = 40x + 30x^2 - 10x^3$ $+16 + 12x - 4x^2$	Simplify.
$(f \cdot g)(x) = (-10x^3) + (30x^2 - 4x^2)$ $+(40x + 12x) + (16)$	Group similar terms together.
$(f \cdot g)(x) = -10x^3 + 26x^2 + 52x + 16$	Combine similar terms.

Try It Yourself!

What is $(f \cdot g)(x)$ if $f(x) = 3x - 2$ and $g(x) = 3x^2 + 2x - 1$?

Example 2: What is $(f \cdot g)(2)$ if $f(x) = 6 - 2x$ and $g(x) = 4 - 3x$?

Solution:

Step 1: Multiply the two given expressions for $f(x)$ and $g(x)$.

$(f \cdot g)(x) = f(x) \cdot g(x)$	
$(f \cdot g)(x) = (6 - 2x)(4 - 3x)$	Substitute the given functions.
$(f \cdot g)(x) = (6)(4) + (6)(-3x)$ $+(-2x)(4) + (-2x)(-3x)$	Multiply the expressions.
$(f \cdot g)(x) = 24 - 18x - 8x + 6x^2$	Simplify.
$(f \cdot g)(x) = 6x^2 + (-18x - 8x) + 24$	Group similar terms together.
$(f \cdot g)(x) = 6x^2 - 26x + 24$	Combine similar terms.

Step 2: Substitute $x = 2$ to the resulting function.

$(f \cdot g)(2) = 6(2)^2 - 26(2) + 24$	Evaluate.
$(f \cdot g)(2) = 24 - 52 + 24$	Simplify.
$(f \cdot g)(2) = -4$	

Try It Yourself!

What is $(g \cdot h)(2)$ if $g(x) = 20 - x^2$ and $h(x) = 5 - x$?

Example 3: Find $(f \cdot g)(x)$, given $f(x) = \frac{x+7}{x+1}$ and $g(x) = \frac{x-2}{x+3}$.

Solution: Multiply the two given expressions for $f(x)$ and $g(x)$.

$(f \cdot g)(x) = f(x) \cdot g(x)$

$(f \cdot g)(x) = \left(\frac{x+7}{x+1}\right)\left(\frac{x-2}{x+3}\right)$ Substitute the given functions.

$(f \cdot g)(x) = \frac{(x+7)(x-2)}{(x+1)(x+3)}$ Multiply the numerator and the denominator.

$(f \cdot g)(x) = \frac{(x)(x)+(x)(-2)+(7)(x)+(7)(-2)}{(x)(x)+(x)(3)+(1)(x)+(1)(3)}$ Multiply the expressions on the numerator and denominator.

$(f \cdot g)(x) = \frac{x^2-2x+7x-14}{x^2+3x+x+3}$ Simplify.

$(f \cdot g)(x) = \frac{x^2+5x-14}{x^2+4x+3}$ Combine similar terms.

Try It Yourself!

Find $(f \cdot g)(x)$, given $f(x) = \frac{x-5}{x-3}$ and $g(x) = \frac{x-1}{x+3}$.

Real-World Problems

Example 4: A farmer has a rectangular plot of land. The length of the plot of land can be expressed as a function $l(x) = x + 2$ meters and its width can be expressed as $w(x) = 3x + 21$ meters. Find the area of the plot.

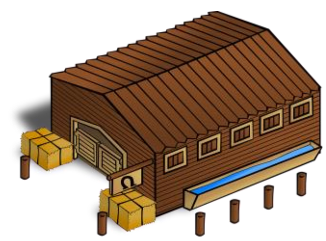
Solution: Use the formula for the area of a rectangle, which is $A = l \cdot w$.

$$(l \cdot w)(x) = l(x) \cdot w(x)$$
$$(l \cdot w)(x) = (x+2)(3x+21)$$
$$(l \cdot w)(x) = (x)(3x) + (x)(21) + (2)(3x) + (2)(21)$$
$$(l \cdot w)(x) = x^2 + 21x + 6x + 42$$
$$(l \cdot w)(x) = x^2 + 27x + 42$$

Thus, the area of the farmer's plot of land is $(x^2 + 27x + 42)$ square meters.

Try It Yourself!

If the plot of land in *Example 4* will be converted into a warehouse, what would be the total volume $V(x)$ of the structure if the height of the walls from the floor to the flat roof is $(x+5)$ meters high.



Check Your Understanding!

1. Consider the following functions:

$a(x) = x + 1$
$b(x) = x^2 - 1$
$c(x) = 2x + 3$
$d(x) = 3x - 5$
$e(x) = 3x^2 + 4$
$f(x) = 5x - 12$
$g(x) = \frac{x^2-1}{3x+5}$
$h(x) = \dfrac{x^2 - 2x + 1}{x+1}$
$i(x) = \dfrac{x^2 - 1}{3x^2 + x - 5}$

Find:

a. $(a \cdot e)(x)$
b. $(c \cdot d)(-2)$
c. $(d \cdot e)(x)$
d. $(f \cdot d)(x)$
e. $(c \cdot e)(3)$
f. $(b \cdot h)(-3)$
g. $(d \cdot i)(x)$
h. $(a \cdot i)(x)$
i. $(g \cdot h)(2)$
j. $(h \cdot i)(-2)$

2. Consider the following functions:

$p(x) = 2x + 4$
$q(x) = x^2 + 2x - 3$
$r(x) = x - 1$
$s(x) = x^2 - 3$
$t(x) = x + 2$
$u(x) = 2x - 1$
$v(x) = x^2 + 2x$

   a. Express $f_1(x) = x^2 + x - 2$ as a product of any two given functions.
   b. Express $f_2(x) = 2x^2 + 2x - 4$ as a product of any two given functions.
   c. Express $f_3(x) = 2x^3 + 8x^2 + 8x$ as a product of any two given functions.
   d. Express $f_4(x) = 2x^3 + 3x^2 - 3x$ as a product of any two given functions.
   e. Express $f_5(x) = x^4 + 2x^3 - 6x^2 - 6x + 9$ as a product of any two given functions.

3. The area of the base of a cylindrical container is defined by $b(x) = x^2 - x + 12$. Find the volume of water that can be stored inside the container if the height is defined by $h(x) = x - 1$.

Lesson 3: Division of Functions

Warm Up!

First Three!

Materials Needed: paper and pen

Instructions:

1. This activity may be done individually or by pair.
2. Your teacher will present a problem on the board. There will be 2 questions per category.

3. For the easy category, you will be asked to answer the problem mentally and only the final answer will be considered. For the average category, you will be asked to show your solution using synthetic division. For the difficult category, you will be asked to solve using long division.
4. The first five persons or pairs to give the correct answer gain a point or incentive.
5. Individuals or pairs who have correctly answered the first question for each category may be exempted to give chance to others.

Sample Questions:
Easy: $(x^2 - 25) \div (x + 5)$
Average: $(x^3 + 3x^2 - 13x - 15) \div (x + 5)$
Difficult: $(2x^4 + 8x^3 - 20x^2 - 56x - 30) \div (2x + 2)$

Learn about It!

Algebraic expressions can be used to represent rules for relations and consequently, functions. In this lesson, we will discuss how to find the quotient of two given functions.

When given two functions $f(x)$ and $g(x)$, their quotient, denoted by $\left(\frac{f}{g}\right)(x)$, is the function defined by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$. By the definition of the quotient of two functions, we just divide the two given expressions for $f(x)$ and $g(x)$.

Let us study the example given below.

Given $f(x) = 5x^3 + 12x^2 + 4x$ and $g(x) = x^2$, find $\left(\frac{f}{g}\right)(x)$.

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$\left(\frac{f}{g}\right)(x) = \frac{5x^3+12x^2+4x}{x^2}$ Substitute the given functions.

$\left(\frac{f}{g}\right)(x) = \frac{x(5x^2+12x+4)}{x^2}$ Factor both the dividend and the divisor

$\left(\frac{f}{g}\right)(x) = \frac{5x^2+12x+4}{x}$ Cancel the common factor/s.

Keep in mind that aside from factoring, there are also other ways to simplify and find quotients of functions like long division and synthetic division.

Let us study some other examples below to better understand division of functions.

Let's Practice!

Example 1: What is $\left(\frac{f}{g}\right)(x)$, if $f(x) = x^2 - 1$ and $g(x) = x + 1$?

Solution: Divide the two given expressions for $f(x)$ and $g(x)$.

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$\left(\frac{f}{g}\right)(x) = \frac{x^2-1}{x+1}$ Substitute the given functions.

$\left(\frac{f}{g}\right)(x) = \frac{(x+1)(x-1)}{x+1}$ Factor the dividend.

$\left(\frac{f}{g}\right)(x) = x - 1$ Divide/ Simplify the common factor.

Try It Yourself!

What is $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, if $f(x) = x^2 + 6x + 5$ and $g(x) = x + 1$?

Example 2: Find $\left(\frac{f}{g}\right)(x)$ if $f(x) = 2x^3 - 7x^2 + 2x + 3$ and $g(x) = x - 3$.

Solution: Divide the two given expressions for $f(x)$ and $g(x)$.

Step 1: Substitute the given functions.

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$
$$\left(\frac{f}{g}\right)(x) = \frac{2x^3 - 7x^2 + 2x + 3}{x - 3}$$

Step 2: We can try factoring the numerator. However, it may be time-consuming so let us try dividing the functions directly. We can use long division or synthetic division to accomplish this.

3	2	−7	2	3
		6	−3	−3
	2	−1	−1	0

Since we have a remainder of zero, we can conclude that the quotient of the two given functions is $2x^2 - x - 1$.

Try It Yourself!

Find $\left(\frac{f}{g}\right)(x)$ if $f(x) = 2x^4 - 5x^3 - 5x^2 + 5x + 3$ and $g(x) = x - 3$.

Example 3: Find $\left(\frac{f}{g}\right)(5)$ if $f(x) = x^3 - 1$ and $g(x) = x^2 - 1$.

Solution:

Step 1: Divide the two given expressions for $f(x)$ and $g(x)$.

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

$$\left(\frac{f}{g}\right)(x) = \frac{x^3-1}{x^2-1}$$ Substitute the given functions.

$$\left(\frac{f}{g}\right)(x) = \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)}$$ Factor the dividend and divisor.

$$\left(\frac{f}{g}\right)(x) = \frac{x^2+x+11}{x+1}$$ Divide/Simplify the common factor.

Step 2: Substitute $x = 5$ to the resulting function.

$$\left(\frac{f}{g}\right)(5) = \frac{(5)^2+5+1}{5+1}$$ Evaluate.

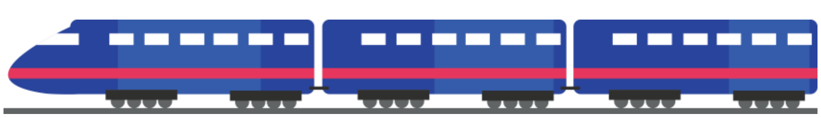
$$\left(\frac{f}{g}\right)(5) = \frac{31}{6}$$ Simplify.

Try It Yourself!

Find $\left(\frac{f}{g}\right)(5)$ if $f(x) = x^3 - 1$ and $g(x) = x^2 - 2x + 1$.

Real-World Problems

Example 4: A train traveling at top speed reached its destination in $t(x) = x - 2$ hours. If the distance traveled by the train is $d(x) = x^2 + 3x - 10$ kilometers, find the top speed $v(x)$ of the train.



Solution: Distance divided by time is equal to speed, denoted by $v(x) = \left(\frac{d}{t}\right)(x) = \frac{d(x)}{t(x)}$. Thus, we can obtain the required speed by dividing the given functions that represent distance and time.

From the problem, we have $d(x) = x^2 + 3x - 10$ and $t(x) = x - 2$.

2	1	3	-10
		2	10
	1	5	0

Therefore, the speed of the train is $(x + 5)$ kilometers per hour.

Try It Yourself!

An electronically controlled printing machine can print a number of words defined by $w(x) = x^4 - x^3 + 2x^2 - x - 1$. If a sheet of paper fed in the printer will only accommodate $p(x) = x - 1$ words, how many sheets of paper can the machine print on? Use the function $s(x)$ to define the number of sheets.

Check Your Understanding!

1. Consider the following functions:

$a(x) = x + 1$

$b(x) = x - 2$

$c(x) = x + 3$

$d(x) = x^2 + 4x + 1$

$e(x) = x^2 - 125$

$f(x) = 5x - 10$

$g(x) = \dfrac{x^2 - 1}{x - 5}$

$h(x) = \dfrac{x^2 + 2x + 1}{x - 2}$

$i(x) = \dfrac{x^2 - 1}{x^2 + x - 6}$

$j(x) = x^3 + 2x^2 - 6x - 2$

Find:

a. $\left(\frac{d}{a}\right)(0)$

b. $\left(\frac{d}{c}\right)(-2)$

c. $\left(\frac{e}{f}\right)(1)$

d. $\left(\frac{d}{a}\right)(-3)$

e. $\left(\frac{j}{a}\right)(3)$

f. $\left(\frac{j}{d}\right)(-3)$

g. $\left(\frac{h}{i}\right)(-2)$

h. $\left(\frac{i}{j}\right)(1)$

i. $\left(\frac{j}{c}\right)(2)$

j. $\left(\frac{j}{b}\right)(-1)$

2. Consider the following functions:

$f(x) = 2x + 1$

$g(x) = 2x - 2$

$h(x) = 2x^2 - 5x - 3$

$p(x) = 2x^3 - 7x^2 + 2x + 3$

$q(x) = x^2 - 2x + 1$

$r(x) = \dfrac{2x+1}{x-1}$

a. Express $f_1(x) = x - 1$ as a quotient of any two given functions.

b. Express $f_2(x) = \frac{1}{x-1}$ as a quotient of any two given functions.

c. Express $f_3(x) = \frac{x-1}{2}$ as a quotient of any two given functions.

d. Express $f_4(x) = x - 3$ a quotient of any two given functions.

e. Express $f_5(x) = \frac{2x^2-5x-3}{2}$ as a quotient of any two given functions.

3. A train travels from Tutuban to Calamba station. The distance between Tutuban and Calamba is defined by the function $d(x) = x^4 + x^3 + 3x^2 + 4x - x$ kilometers. If the train travels from Tutuban to Calamba for $t(x) = x$ hours without stopping at any station, how fast is the train traveling?

Lesson 4: Composition of Functions

Warm Up!

Input-Process-Output

Materials Needed: paper and pen, marker and cartolina, drawing and coloring materials

Instructions:

1. This activity may be done individually or by group.
2. The teacher will let you roam around the school to observe or imagine your day-to-day activities. Recall all activities that involve processes wherein there is an input material and an output product.
3. You will be given time to make an illustration regarding the said process focusing on the flow: INPUT-PROCESS-OUTPUT.
4. A volunteer will be called to show and briefly explain their work.

Learn about It!

The activity in *Warm Up!* is a great way on how to envision functions in general. It functions the same way as other machines that you may encounter in real life. As what we have elaborated in the previous activity, some machines require an input in their process that will result in a specific output. If you put dirty clothes in a washing machine and let it tumble and wash, then you will have washed clothes.

The same idea is also present when we evaluate functions in terms of the value of x. However, how will we proceed with our solution if the input to our function is another function?

The process of combining functions, where the output of one function is used as the input of the other function, is known **as function composition**. Given two functions $f(x)$ and $g(x)$, their composite is denoted by $(f \circ g)(x) = f(g(x))$. It is read as "*f composed with g of x" or "f of g of x*".

To have a better understanding of composition of functions, let's study the examples below.

Given two functions $f(x) = 5x + 2$ and $g(x) = x^2$, find $(f \circ g)(x)$.

$(f \circ g)(x) = f(g(x))$	
$(f \circ g)(x) = f(x^2)$	Substitute the inner function.
$(f \circ g)(x) = 5(x^2) + 2$	Evaluate the main function in terms of the inner function.
$(f \circ g)(x) = 5x^2 + 2$	Simplify.

Notice that if there are more terms in the inner function, the main function gains more terms.

Let's Practice!

Example 1: What is $(f \circ g)(x)$ if $f(x) = 5x^3 + 12x^2 + 4x$ and $g(x) = x^2$.

Solution:

$(f \circ g)(x) = f(g(x))$	
$(f \circ g)(x) = f(x^2)$	Substitute the given input function.

$(f \circ g)(x) = 5(x^2)^3 + 12(x^2)^2 + 4(x^2)$ Evaluate in terms of the input function.

$(f \circ g)(x) = 5x^6 + 12x^4 + 4x^2$ Simplify.

Try It Yourself!

What is $(f \circ g)(x)$ if $f(x) = 2x^3 - 4x^2 + 12x + 3$ and $g(x) = 4x$?

Example 2: What is $(g \circ f)(x)$ if $f(x) = 5x^3 + 12x^2 + 4x$ and $g(x) = x^2 + 6$.

Solution:

$(g \circ f)(x) = g(f(x))$

$(g \circ f)(x) = g(5x^3 + 12x^2 + 4x)$ Substitute.

$(g \circ f)(x) = (5x^3 + 12x^2 + 4x)^2 + 6$ Evaluate and simplify.

$(g \circ f)(x) = 25x^6 + 120x^5 + 184x^4$
$+96x^3 + 16x^2 + 6$ Simplify.

Try It Yourself!

What is $(f \circ g)(x)$ if $f(x) = x^3 - x^2 + 4$ and $g(x) = x + 1$?

Example 3: Find $(f \circ g)(2)$ given $f(x) = 2x^3 - 4x^2 - 5$ and $g(x) = x^2$.

Solution:

$(f \circ g)(x) = f(g(x))$

$(f \circ g)(x) = f(x^2)$ Substitute the given input function.

$(f \circ g)(x) = 2(x^2)^3 - 4(x^2)^2 - 5$ Evaluate in terms of the input function.

$(f \circ g)(x) = 2x^6 - 4x^4 - 5$ — Simplify.

$(f \circ g)(2) = 2(2)^6 - 4(2)^4 - 5$ — Evaluate in terms of the given value of x.

$(f \circ g)(2) = 128 - 64 - 5$ — Simplify.

$(f \circ g)(2) = 59$

Try It Yourself!

What is $(f \circ g)(4)$ if $f(x) = x^2 + 3x - 12$ and $g(x) = x + 2$.

Real-World Problems

Example 4: Angelo works 45 hours a week at a printing press. He receives a weekly salary of ₱1500 plus 2% commission for every 1000 printouts. Assuming that Angelo makes enough printouts this week to earn his commission, what is the composition that represents his commission, given the functions $P(x) = 0.02x$ and $R(x) = x - 1000$?

Solution: The composition $(P \circ R)(x)$ means to take the number of printouts x, subtract the 1000 printouts, and then multiply by 2%.

$(P \circ R)(x) = P(R(x))$

$(P \circ R)(x) = P(x - 1000)$ — Substitute the given input function.

$(P \circ R)(x) = 0.02(x - 1000)$ — Evaluate in terms of the input function.

$(P \circ R)(x) = 0.02x - 20$ — Simplify.

Thus, the composite function for his commission is $(P \circ R)(x) = 0.02x - 20$.

Try It Yourself!

The radius of a spherical balloon inflates at a rate of $r(t) = 3t$ centimeters per t seconds. What will be the total increase in volume of the balloon after t seconds? How large is the balloon from one side to the other side if it can be totally inflated in 5 seconds? (Hint: $V = \frac{4}{3}\pi r^3$)

Check Your Understanding!

1. Consider the following functions:

$a(x) = 2x - 1$

$b(x) = 3x$

$c(x) = x^2 - 1$

$d(x) = x^2 - 2x + 1$

$e(x) = \dfrac{x+1}{2x}$

$f(x) = \dfrac{x^2 - x + 5}{x - 2}$

Find:

a. $a(b(x))$

b. $b(c(x))$

c. $d(c(x))$

d. $c(a(x))$

e. $d(b(x))$

f. $e(b(x))$

g. $e(a(x))$

h. $c(d(x))$

i. $a(e(x))$

j. $f(c(x))$

2. Consider the following functions:

$f(x) = 9 - x$ $\qquad$ $g(x) = x^2 - x$ $\qquad$ $h(x) = x + 2$

a. Evaluate $(f \circ h)(-3)$.

b. Evaluate $(h \circ f)(2)$.

c. Evaluate $(g \circ f)(1)$.

d. Evaluate $(f \circ g)(-2)$.

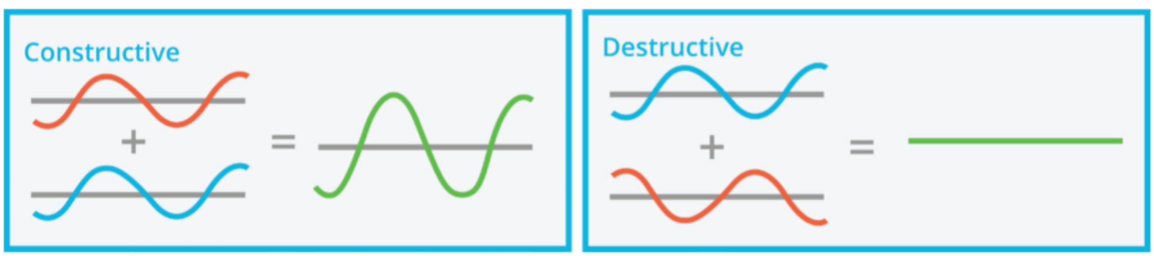
e. Evaluate $(f \circ f)(3)$.

f. Evaluate $(g \circ h)(5)$.

3. Every person needs to pay taxes due to him or her. Suppose a freelance editor is accountable to pay 10% tax each month. Jill, a freelance editor, earns x pesos daily for 20 days each month. How much tax will she pay? Express your answer as a composition of functions.

Challenge Yourself!

1. In physics, waves may combine constructively or destructively, as shown in the following diagrams.



Write two functions $f(x)$ and $g(x)$ (do not choose linear functions). Then, calculate their sum $(f+g)(x)$ and difference $(f-g)(x)$. Graph all of these functions. How do these graphs compare with the diagrams of constructive and destructive interference?

2. The following table shows the average price per kilogram of fish, chicken, and beef for the years 2013 to 2015, as well as the total amount of each type consumed by a family (in kilograms).

Year	Price of Fish	Quantity of Fish	Price of Chicken	Quantity of Chicken	Price of Beef	Quantity of Beef
2013	₱80	40	₱250	22	₱270	18
2014	₱85	55	₱280	25	₱300	25
2015	₱90	50	₱300	28	₱320	27

How much money was spent on all the food described in the table? Create a model to describe the expenditure for each year. Compare the differences between the years on the table given.

3. "All functions are relations but not all relations are functions". Explain briefly the context of the given statement, and give 5 examples using the concepts you have learned in this unit to support your answer. Show your complete solution.

4. Suppose $N(x) = x$ is a function that defines the total number of shoes sold at a department store, $p(x) = 1200 - 4x$ is the price of shoes bought, for $0 \leq x \leq 20$, and $d(x) = 0.2x$ is the discount given during a sale.
   a. What function will represent the total revenue earned?
   b. What function will represent the discount for items sold at a discounted price?
   c. What function will represent the total revenue earned if there is a sale?

5. Let x represent the price of a regular laptop computer.
   a. Give a function f that will represent the price of the laptop when a 1000 pesos price reduction is applied.
   b. Give a function g that will represent the price of the laptop when a 10% discount will be given.
   c. Solve for $f(g(x))$ and $g(fx))$. What do these two composite functions represent? Which of the two is more beneficial to the customer? Explain briefly.

Performance Task

As a Senior High School student, you will be encountering three different types of subjects, the CORE, SPECIALIZED, and APPLIED subjects. One of the core subjects you will study alongside General Mathematics is Disaster Readiness and Risk Reduction.

As a collaborative approach, your teacher in General Mathematics and DRRR asked your class to prepare a research proposal integrating the concepts about environmental preservation and operations and composition of functions. The preliminary paper to be passed should contain an introduction and mathematical justifications of real-life models of functions based on and related to the stated environmental preservation topics written on the introduction.

You are required to pass a 2-page introduction, with 1.5-line spacing, font size of 12, and a font style of Arial Narrow. A mathematical justification part will follow the introduction containing, but not limited to, ten real-life word problems or situations modeling the concepts of operations on functions and composition of functions together with their respective solutions. The solutions for the word problems must also be included in this part. An initial title may also be included but not required. You may be asked to write your research proposal individually or by groups of four.

Your research proposal will be evaluated by the two teachers based on the following criteria in the given performance task rubric as follows.

The top 5 individuals or groups with the highest points will be asked to present their work in a mini class research symposium to highlight the relevance of environmental preservation.

Performance Task Rubric

Criteria	Below Expectation (0–49%)	Needs Improvement (50–74%)	Successful Performance (75–99%)	Exemplary Performance (99+%)
Mathematical Justification	Justification is ambiguous. Only few concepts learned in the unit are applied.	Justification are not so clear. Some ideas are not connected to each other. Not all concepts learned in the unit are applied.	Justification is clear and informatively delivered. Appropriate concepts learned in the unit are applied.	Justification is logically clear, informative, and professionally delivered. The concepts learned in the unit are applied.

Accuracy	The information given are erroneous and do not show wise use of the concepts from the integrated subjects.	The information given are erroneous and show some use of the concepts from the integrated subjects.	The information given are accurate and shows use of the concepts from the integrated subjects.	The information given are accurate and show a wise use of the concepts from the integrated subjects.
Impact	The introduction is poor.	The introduction is somewhat informative.	The introduction is informative and flawless, with minimal errors.	The introduction is very informative and flawlessly done. It is also easily understandable.
Efficiency	The introduction has more than 10 grammatical, capitalization, spelling, or any errors.	The introduction has 6 to 10 grammatical, capitalization, spelling, or any errors.	The introduction has 1 to 5 grammatical, capitalization, spelling, or any errors.	The introduction has no grammatical, capitalization, spelling, or any errors.

Wrap-up

Operation	Formula
Addition	$(f+g)(x)=f(x)+g(x)$
Subtraction	$(f-g)(x)=f(x)-g(x)$

Multiplication	$(f \cdot g)(x) = f(x) \cdot g(x)$
Division	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$
Composition	$(f \circ g)(x) = f\big(g(x)\big)$

Key to *Let's Practice!*

Lesson 1

1. $(g + h)(x) = -3x^4 + x^3 + 8x^2 + 2x - 3$
2. $(f - g)(x) = 72$
3. $(f - g)(x) = \frac{11x+23}{x^2+4x+3}$
4. $(d - c)(5) = 5397$. Therefore, he has ₱5,397 left for his allowance.

Lesson 2

1. $(f \cdot g)(x) = 9x^3 - 7x + 2$
2. $(g \cdot h)(2) = 48$
3. $(f \cdot g)(x) = \frac{x^2-6x+5}{x^2-9}$
4. The total volume of the structure is defined by $V(x) = x^3 + 32x^2 + 177x + 210$

Lesson 3

1. $\left(\frac{f}{g}\right)(x) = x + 5$
2. $\left(\frac{f}{g}\right)(x) = 2x^3 + x^2 - 2x - 1$
3. $\left(\frac{f}{g}\right)(5) = \frac{31}{4}$
4. A total of $(x^3 + 2x + 1)$ sheets are needed.

Lesson 4

1. $(f \circ g)(x) = 128x^3 - 64x^2 + 48x + 3$
2. $(f \circ g)(x) = x^3 + 2x^2 + x + 4$
3. $(f \circ g)(4) = 42$
4. The volume of the balloon will be $V(x) = 4500\pi$ or $V(x) \approx 14137.16694$ cubic centimeters.

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